

# Asymptotic Iteration Method for singular potentials

Brodie Champion

*Department of Mathematics and Statistics,  
University of Prince Edward Island,  
550 University Avenue, Charlottetown,  
Prince Edward Island, Canada C1A 4P3  
bchampion@upei.ca*

Richard L Hall

*Department of Mathematics and Statistics, Concordia University,  
1455 de Maisonneuve Boulevard West,  
Montréal, Québec, Canada H3G 1M8  
rhall@mathstat.concordia.ca*

Nasser Saad

*Department of Mathematics and Statistics,  
University of Prince Edward Island,  
550 University Avenue, Charlottetown,  
Prince Edward Island, Canada C1A 4P3  
nsaad@upei.ca*

The asymptotic iteration method (AIM) is applied to obtain highly accurate eigenvalues of the radial Schrödinger equation with the singular potential  $V(r) = r^2 + \lambda/r^\alpha$  ( $\alpha, \lambda > 0$ ) in arbitrary dimensions. Certain fundamental conditions for the application of AIM, such as a suitable asymptotic form for the wave function, and the termination condition for the iteration process, are discussed. Several suggestions are introduced to improve the rate of convergence and to stabilize the computation. AIM offers a simple, accurate, and efficient method for the treatment of singular potentials such as  $V(r)$  valid for all ranges of coupling  $\lambda$ .

PACS Nos.: 03.65.Ge.

## I. INTRODUCTION

Attractive potentials with a strong repulsive core are common in atomic, nuclear and molecular physics [1]–[13]. A typical class of such potentials [4]–[22] are the spiked harmonic oscillators  $V(r) = r^2 + \frac{\lambda}{r^\alpha}$ . The potential  $V(r)$  is called ‘spiked’ because of its pronounced peak near the origin. For  $\alpha > 2$ , the potential is of relevance to quantum field theory, describing so called supersingular interactions for which matrix elements of the perturbation in the harmonic-oscillator basis diverge: thus every term in the perturbation series is infinite, and the perturbation expansion does not exist [3]. There are several other reasons for interest in this class of potentials. First, it represents the simplest example of certain class of realistic interaction models in atomic, nuclear and molecular physics. Second, in the one-dimensional case, the perturbed-oscillator operator  $H = p^2 + V(r) = -d^2/dr^2 + r^2 + \lambda/r^\alpha = H_0 + \lambda/r^\alpha$ , where  $p = -i\partial/\partial r$ , may not converge to the original one  $H_0$  as  $\lambda \rightarrow 0$  (The Klauder phenomenon). Third, the perturbation series is ordered in fractional power [4, 14] in  $\lambda$ . Since the early study of Harrell on singular perturbation theory for the ground-state energy of the Hamiltonian  $H$ , an enormous amount of work has been done to investigate the spectral problems of this operator. Most of the work, however, is either devoted to the study of the ground-state energy for  $\lambda$  near zero, which represents the most challenging problem, or for particular values of the potential parameters. Owing to the difficulties inherited by the potential structure near the origin, much attention must be paid to the selected method for tackling these problems. The purpose of this letter is twofold. First to develop a simple and easily adopted technique, based on the asymptotic iteration method [22], to compute the eigenvalues of the radial

Schrödinger equation

$$-\frac{d^2\psi(r)}{dr^2} + \left(r^2 + \frac{\gamma(\gamma+1)}{r^2} + \frac{\lambda}{r^\alpha}\right)\psi(r) = E\psi(r), r \in [0, \infty), \quad (1)$$

where  $\alpha > 0$ , and the eigenfunctions  $\{\psi(r)\}$  satisfy the Dirichlet boundary condition  $\psi(0) = 0$ . This is valid for the one-dimensional case as well as for higher dimensions  $N > 1$  through  $\gamma = l + \frac{1}{2}(N - 3)$ , regardless of the values of the potential parameters. Second, to point out the importance of the correct form of asymptotic wave function to stabilize the iteration technique, and also to provide some suggestions to improve the rate of convergence of AIM when it is used to tackle Schrödinger eigenvalue problems with a wide variety of other singular potentials of physical and chemical interest. In order to achieve these goals, we develop first a wave function with the right exponential tail and which satisfies the Dirichlet boundary condition at the origin. This optimizes and stabilizes the use of AIM for computing the eigenvalues. The asymptotic iteration method (AIM) was originally introduced [22] to investigate the solutions of differential equations of the form

$$y'' = \lambda_0(r)y' + s_0(r)y, \quad \left(' = \frac{d}{dr}\right) \quad (2)$$

where  $\lambda_0(r)$  and  $s_0(r)$  are  $C^\infty$ -differentiable functions. Using AIM, the differential equation (2) has a general solution [22]:

$$y(r) = \exp\left(-\int^r \rho(t)dt\right) \left[ C_2 + C_1 \int^r \exp\left(\int^t (\lambda_0(\tau) + 2\rho(\tau))d\tau\right) dt \right] \quad (3)$$

where, for sufficiently large  $n > 0$ , we obtain the  $\rho(r)$  values from

$$\frac{s_n(r)}{\lambda_n(r)} = \frac{s_{n-1}(r)}{\lambda_{n-1}(r)} \equiv \rho(r) \quad (4)$$

for

$$\begin{aligned} \lambda_n &= \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1} \\ s_n &= s'_{n-1} + s_0 \lambda_{n-1}, \quad n = 1, 2, 3, \dots \end{aligned} \quad (5)$$

It should be noted that one can start the iteration from  $n = 0$  with the initial condition  $\lambda_{-1} = 1$  and  $s_{-1} = 0$ . Since Ref.[22] the method has been adopted to investigate the spectrum of many problems in relativistic and non-relativistic quantum mechanics [22]–[33]. In the process of applying AIM, especially in its application to eigenvalue problems of Schrödinger-type, such as (1), one usually faces the following two problems.

#### A. Asymptotic solution Problem:

The first problem we are confronted with in applying AIM is the conversion of the eigenvalue problem (the absence of first derivative) to standard form suitable to utilize AIM (2). A general strategy to overcome this problem is to use an asymptotic solution  $\psi_a(r)$  which satisfies the boundary conditions of the given eigenvalue equation. By substitution of the assumed exact solution with the form  $\psi(r) = \psi_a(r)f(r)$  into the eigenvalue problem, once the  $\lambda_0$  and  $s_0$  have been determined, the sequences  $\lambda_n$  and  $s_n$  can be computed using (5). The energy eigenvalues are then obtained from the roots of the termination condition (4), which can be written in more convenient form as

$$\delta_n(r; E) = \lambda_n(r; E)s_{n-1}(r; E) - \lambda_{n-1}(r; E)s_n(r; E) = 0, \quad n = 1, 2, \dots \quad (6)$$

For Schrödinger's equation (1), the asymptotic solution is dominated by the harmonic oscillator wave function, since, for larger  $r$ , the dominant term of the potential is the harmonic oscillator term  $r^2$ . The problem with such asymptotic solution, however, is that the behavior of the wave function near the origin has not been considered.

#### B. Termination Condition Problem:

The second problem results when the eigenvalue problem (now in the standard form for AIM application) fails to be exactly solvable. Indeed, if the eigenvalue problem has exact analytic solutions, the termination condition (6) leads

to an expression that depends only on the eigenvalues  $E$ , that is to say, independent of  $r$ . In such cases, the energy eigenvalues are simply the roots of  $\delta_n(E) = 0, n = 1, 2, \dots$ . For example, if  $\alpha = 2$ , Eq.(1) is exactly solvable, and the termination condition (6) yields

$$\delta_n(E) = \prod_{i=0}^n (4i + 3 + 2\gamma - E), \quad n = 0, 1, 2, 3, \dots \quad (7)$$

The condition  $\delta_n(E) = 0$  leads to the exact solutions  $E_n = 4n + 3 + 2\gamma, n = 0, 1, 2, \dots$  as expected [22]. If the eigenvalue problem is not analytically solvable with the analytic form chosen, as for  $0 < \alpha \neq 2$ , then the termination condition (6) produces for each iteration an expression that depends on both  $r$  and  $E$ . In such a case, one faces the problem of finding the best possible starting value  $r = r_0$  that stabilizes the process so that it doesn't oscillate but converges. In principle, the computation of the roots of  $\delta_n(r_0; E) = 0$  should be independent of the choice of  $r_0$ , nevertheless, the right choice of  $r$ , as we shall show in the present work, usually accelerates the rate of convergence to accurate eigenvalues  $E$  within a reasonable number of iterations. Generally, a suitable  $r_0$  value is determined either as the location of the maximum of value of the asymptotic wave function, or as the position of the minimum value of the potential under consideration. A more general and systematic way to choose a suitable value for  $r_0$  is still open question for further research.

For the Schrödinger equation (1), we face these two problems. In the next section we develop an asymptotic wave function that satisfies the boundary conditions at zero and infinity. This asymptotic form is then used in section 3 to initialize the asymptotic iteration method. In section 4, using a suitable value of  $r_0$ , we exhibit and discuss the numerical results of AIM for a wide range of  $\lambda$  and  $\alpha > 0$ . Finally, in section 5, we comment on these results.

## II. ASYMPTOTIC WAVEFUNCTION FOR SINGULAR POTENTIALS

For small  $r$ , one can neglect in (1) the energy  $E$  and the harmonic oscillator term as compared with the perturbative term  $\lambda/r^\alpha$ . Hence near the origin, (1) can be written as

$$-\frac{d^2\psi}{dr^2} + \left( \frac{\gamma(\gamma+1)}{r^2} + \frac{\lambda}{r^\alpha} \right) \psi = 0. \quad (8)$$

Using the transformation

$$\psi(r) = \sqrt{r}\phi(t), \quad t = \beta r^\sigma$$

a straightforward calculation shows that Eq.(8) can be written as

$$\frac{1}{4r^{3/2}}\phi(t) - \frac{\sigma^2\beta}{r^{-\sigma+3/2}}\frac{d\phi}{dt} - \frac{\sigma^2\beta^2}{r^{-2\sigma+3/2}}\frac{d^2\phi}{dt^2} + \frac{\gamma(\gamma+1)}{r^{3/2}}\phi(t) + \frac{\lambda}{r^{\alpha-1/2}}\phi(t) = 0$$

or, in more compact form, as

$$\frac{d^2\phi}{dt^2} + \frac{1}{t}\frac{d\phi}{dt} - \left[ \frac{(2\gamma+1)^2}{4\sigma^2t^2} + \frac{\lambda}{\sigma^2\beta^2}r^{-\alpha+2-2\sigma} \right] \phi = 0.$$

Therefore, for  $2\sigma = 2 - \alpha$  and  $\beta = \sqrt{\lambda}/\sigma$ , we obtain the modified Bessel's differential equation

$$\frac{d^2\phi}{dt^2} + \frac{1}{t}\frac{d\phi}{dt} - \left[ \frac{\nu^2}{t^2} + 1 \right] \phi = 0, \quad (9)$$

where  $\nu = \frac{2\gamma+1}{2-\alpha}$ . The general solution of this differential equation is [34]

$$\phi(t) = c_1 I_\nu(t) + c_2 K_\nu(t),$$

where  $c_1$  and  $c_2$  are constants and  $I_\nu$  and  $K_\nu$  are the modified Bessel functions of the first and second kind respectively [34]. Consequently, an asymptotic solution of (1) for zero energy is given by

$$\psi(r) = \sqrt{r}[c_1 I_\nu(\beta r^\sigma) + c_2 K_\nu(\beta r^\sigma)]. \quad (10)$$

We may now consider two cases, depending on the value of  $\alpha$ .

**Case I** ( $\alpha > 2$ ): In this case  $\sigma < 0$ , the boundary condition  $\psi(0) = 0$  forces  $c_1 = 0$ , hence

$$\psi_a(r) \equiv c_2 \cdot \sqrt{r} K_{\frac{2\gamma+1}{\alpha-2}} \left( \frac{2\sqrt{\lambda}}{\alpha-2} r^{1-\frac{\alpha}{2}} \right). \quad (11)$$

From the asymptotic approximation [34] of  $K_\nu(z)$ , we know for large argument  $z$  that  $K_\nu(z) \equiv e^{-z}/\sqrt{\frac{2z}{\pi}}$ . Therefore, since  $1 - \frac{\alpha}{2} < 0$ , we have for small  $r$ , that

$$\psi_a(r) \equiv c_1 \cdot \frac{1}{2} \sqrt{\frac{\pi(\alpha-2)}{\sqrt{\lambda}}} r^{\alpha/4} e^{-\frac{2\sqrt{\lambda}}{\alpha-2} r^{1-\frac{\alpha}{2}}}. \quad (12)$$

Consequently, for  $\alpha = 2m + 2$ ,  $m > 0$ ,

$$\psi_a(r) \equiv r^{(m+1)/2} e^{-\frac{\sqrt{\lambda}}{m r^m}}, \quad m > 0 \quad (13)$$

up to a constant. In particular, if  $\alpha = 4$ , (i.e.  $m = 1$ ), one recovers the familiar limiting form of the solution when the repulsive potential is proportional to  $r^{-4}$ :

$$\psi_a(r) \equiv r e^{-\frac{\sqrt{\lambda}}{r}} \quad \text{as } r \rightarrow 0.$$

**Case II** ( $0 < \alpha < 2$ ): In this case  $\sigma > 0$ , the boundary condition  $\psi(0) = 0$  forces  $c_2 = 0$  in (10), thus

$$\psi_a(r) \equiv c_2 \cdot \sqrt{r} I_{\frac{2\gamma+1}{2-\alpha}} \left( \frac{2\sqrt{\lambda}}{2-\alpha} r^{1-\frac{\alpha}{2}} \right). \quad (14)$$

For small argument  $z$ , the asymptotic approximation [34] of  $I_\nu(z) \equiv \left(\frac{z}{2}\right)^\nu / \Gamma(\nu+1)$  yields

$$\psi_a(r) \equiv r^{\gamma+1} \quad (15)$$

up to a constant.

### III. APPLICATIONS

Depending on the degree of singularity of the potential at the origin, that is characterized by the positive parameter  $\alpha$ , we have the following two cases:

#### A. The case $\alpha = 2m + 2$ , $m > 0$ :

In this case the Hamiltonian operator

$$H = -\frac{d^2}{dr^2} + r^2 + \frac{\gamma(\gamma+1)}{r^2} + \frac{\lambda}{r^\alpha}, \quad \alpha > 2 \quad (16)$$

leads to a non-Fuchsian singularity [5] at  $r = 0$  of the Schrödinger equation  $H\psi = E\psi$ , because the potential term  $\lambda/r^\alpha$  has a pole of order  $> 2$ . The asymptotic wave function developed in the previous section suggests that the exact solution of (1), in the case  $\alpha > 2$ , takes the form

$$\psi(r) = r^{\frac{m+1}{2}} e^{-\frac{r^2}{2} - \frac{\sqrt{\lambda}}{m r^m}} f(r). \quad (17)$$

The first exponential term in (17) takes into account the fact that for large  $r$ , the term  $r^2$  in (16) dominates over all other terms of the potential, including, of course,  $\gamma(\gamma+1)/r^2$ . For this wave function, Schrödinger's equation (1) now reads

$$f''(r) = \left( 2r - \frac{2\sqrt{\lambda}}{r^{m+1}} - \frac{1+m}{r} \right) f'(r) + \left( 2 + m + \frac{2\sqrt{\lambda}}{r^m} + \frac{(2\gamma+1-m)(2\gamma+1+m)}{4r^2} - E \right) f(r), \quad (18)$$

which is now amenable to AIM applications. Here, the primes of  $f(r)$  in (18) denote the derivatives with respect to  $r$ .

**B. The case  $\alpha = 2m + 2$ ,  $-1 < m < 0$ :**

In this case, the exact solution of (1) assumes, for  $\alpha < 2$ , the form

$$\psi(r) = r^{\gamma+1} e^{-\frac{r^2}{2}} f(r), \quad (19)$$

where again the exponential term in (19) takes into account that for large  $r$ , the term  $r^2$  dominates over all other terms of the potential. In this case, Schrödinger's equation (1) reads

$$f''(r) = 2 \left( r - \frac{\gamma+1}{r} \right) f'(r) + \left( 2\gamma + 3 + \frac{\lambda}{r^\alpha} - E \right) f(r), \quad (20)$$

which is suitable for an AIM application.

**IV. ITERATIVE SOLUTIONS**

For a given  $\alpha$ , using (18) or (20), we can explicitly write  $\lambda_0(r)$  and  $s_0(r)$  as:

- $\alpha > 2$ :

$$\begin{cases} \lambda_0(r) = \left( 2r - \frac{2\sqrt{\lambda}}{r^{m+1}} - \frac{1+m}{r} \right), \\ s_0(r) = \left( 2 + m + \frac{2\sqrt{\lambda}}{r^m} + \frac{(2\gamma+1-m)(2\gamma+1+m)}{4r^2} - E \right) \end{cases} \quad (21)$$

where  $m = \frac{1}{2}(\alpha - 2)$ .

- $\alpha < 2$ :

$$\begin{cases} \lambda_0(r) = 2 \left( r - \frac{\gamma+1}{r} \right), \\ s_0(r) = \left( 2\gamma + 3 + \frac{\lambda}{r^\alpha} - E \right) \end{cases} \quad (22)$$

and, by means of the iteration formulas (5), we calculate  $\lambda_n(r)$  and  $s_n(r)$ ,  $n = 1, 2, \dots$ . The eigenvalues are then computed by means of the termination condition (6), namely  $\delta_n(r; E) = 0$ . With several symbolic mathematical

TABLE I: The effect of using different values of  $r = r_0$  on computing the eigenvalues using AIM for the Schrödinger equation  $-\frac{d^2\psi}{dr^2} + \left(r^2 + \frac{0.1}{r^4}\right)\psi = E\psi$ . An accurate value (to seven figures) is  $E_{\text{exact}} = 3.575\,552$ . Here,  $N$  refers to the number of iterations needed to achieve such accuracy.

	$r_0$				
$N$	1	2	3	4	5
15	3.478854	3.574644	3.570009	3.561477	3.555018
20	Fails	3.575335	3.573786	3.569455	3.562441
40	“”	3.575551	3.575505	3.575298	3.574726
60	“”	Fails	3.575549	3.575529	3.575458
85	“”	“”	3.575552	3.575550	3.575542
90	“”	“”	Done	3.575551	3.575545
115	“”	“”	“”	3.575552	3.575551
120	“”	“”	“”	Done	3.575551

programs available (*Maple*, *Mathematica*, etc), the computation of the eigenvalues by means of the iteration method, provided it is set up correctly, is a straightforward calculation, even for the higher iteration steps. Most of our computations in the present work were done using *Maple* version 9 running on an IBM architecture personal computer (Dell Dimension 4400). As we mentioned above, the computation of the eigenvalues by means of (6) should be independent of the choice of  $r$ . However, in some applications, for certain values of  $r$ , we may encounter oscillations

of the computed roots that seem to diverge in behavior. This is presumably due to rounding and computational errors in the algorithms used. In Table 1, we show the effects of choosing different starting values of  $r \equiv r_0$  on the number of iterations. It is clear that  $r_0 \geq 3$  is sufficient as starting value of  $r$ . However, suitable choices for  $r_0$  can significantly reduce the number of iterations needed to achieve the required accuracy. In many cases, we have removed the oscillating behavior by increasing the number of significant digits that *Maple* uses in numerical computations. In Table 2, we illustrate the effect of using different numbers of significant digits on the iteration convergence using a *Maple* environment. As indicated by the results in the table, a higher-precision environment can remove the oscillation behavior, as well as stabilize the numerical computation of the root problem by means of the iteration process. In order to accelerate the computation we have written our code for root-finding algorithm instead of using the default procedure *Solve* of *Maple*. For values of  $\alpha = 1$ , AIM gives excellent results, even for extremely small value of the

TABLE II: The effect of using different number of digits (in *Maple 9*) when computing the eigenvalues using AIM for the Hamiltonian  $-\frac{a^2}{dr^2} + r^2 + \frac{6}{r^2} + \frac{0.1}{r^3}$ . Here we set  $r = 3$  in (6). An accurate value is  $E_{\text{exact}} = 7.029816$ . Here  $N$  refers to the number of iterations.

	Number of Digits			
$N$	10	14	18	22
70	7.0301659550	7.0298160740	7.0298162024	7.0298162024
75	7.0298754200	7.0298162383	Done	Done
80	7.0298152350	7.0298162511	“”	“”
85	7.0298160500	7.0298162636	“”	“”
90	7.0298162400	Done	“”	“”

coupling parameter  $\lambda$ . In table 3, we report the AIM results for a considerable range of  $\lambda$  values. These computations have been made with  $r_0 = 3$ . For fractional  $\alpha$ , such as  $1/2, 3/2, 1.9$  etc., the method is still stable and works well; however the number of iterations is much larger than that needed for integer  $\alpha$ . The eigenvalues reported in Table 3 are in excellent agreement with the exact eigenvalue computed by means of a numerical integration of Schrödinger's equation.

TABLE III: The ground-state energy of Schrödinger's equation (1) for  $\alpha = 1$  and different values of  $\lambda$  using the present work. For these computations  $r_0 = 3$ .  $N$  is the number of iterations.

$\alpha = 1$		
$\lambda$	$E_{AIM}$	N
1000	190.723 307 439 784 825 395 54	90
100	42.462 918 114 619 200 840 54	32
10	10.577 483 539 371 157 357 99	52
1	4.057 877 007 967 971 192 93	64
0.1	3.112 066 906 502 466 751 74	65
0.01	3.011 276 010 524 898 166 93	62
0.001	3.001 128 301 284 079 220 13	60
0.0001	3.000 112 837 137 807 781 38	56
0.00001	3.000 011 283 783 881 865 84	55
0.000001	3.000 001 128 379 089 204 58	54

In Table 4 we report the AIM results for the case  $\alpha = 4$ , again for considerable range of  $\lambda$ . The large number of iteration for small values of  $\lambda$  reflect the stability of the method and also the applicability of AIM to treat such cases in one single formalism. Similar tables can be easily constructed for  $\alpha = 3, 5, 6$ , etc. Fewer numbers of iterations are usually needed to achieve any required accuracy for the cases  $\gamma > 0$ . For the ground state energy with  $\gamma = 0$ , much attention to the value of  $r_0$  must be paid to obtain accurate eigenvalues. An important observation is that: for large  $n$ , the computed roots by means of the termination condition (6) are either in descending order or in ascending order; if an oscillation is observed, which appears to change the order, then the starting  $r_0$  should be revised accordingly. The main point is that with the proper choice of  $r_0$ , AIM is a stable and efficient method to obtain the eigenenergies to any degree of accuracy.

## V. CONCLUSION

The present work points out the importance of the asymptotic wave function used for initializing AIM sequences for Schrödinger eigenvalue problems. By introducing a wave function form that satisfies both boundary conditions at zero and at infinity, we were able to obtain accurate eigenvalues for the Schrödinger equation with singular potentials. Several suggestions are discussed to remove the numerical instabilities that may be encountered with direct utilization of AIM. Although we have focused our attention on the calculation of eigenenergies, the method also yields the corresponding eigenfunctions via equations (3) and (4).

## Acknowledgments

Partial financial support of this work under Grant Nos. GP3438 and GP249507 from the Natural Sciences and Engineering Research Council of Canada is gratefully acknowledged by two of us (respectively [RLH] and [NS]).

- 
- [1] Case K M 1971 *Phys. Rev.* **80** 797-806.
  - [2] Frank W M, Land D J and Spector R M 1971 *Rev. Mod. Phys.* **43** 36-98.
  - [3] L C Detwiler and J R Klauder, *Phys. Rev. D* **11** 1436 (1975).
  - [4] Harrell E M 1977 *Ann. Phys. (N.Y.)* 105 379-406.
  - [5] Esposito G 2000 *Foundation of Physics letters* **13** 29-40.
  - [6] Klauder J R, *Acta Phys. Austriaca (Suppl.)* **11**, 341 (1973).
  - [7] B Simon, *J. Funct. Anal.* **14** 295 (1973).
  - [8] B DeFacio and C L Hammer, *J. Math. Phys.* **15** 1071 (1974).
  - [9] L C Detwiler and J R Klauder, *Phys. Rev. D* **11** 1436 (1975).
  - [10] H. Ezawa, J. R. Klauder, and L. A. Shepp, *J. Math. Phys.* **16** 783 (1975).
  - [11] Aguilera-Navarro V C and Guardiola R 1991 *J. Math. Phys.* **32** 2135-2141.
  - [12] Estévez-Bretón E S and Estévez-Bretón G A 1993 *J. Math. Phys.* **34** 437.
  - [13] Znojil M 1993 *J. Math. Phys.* **34** 4914.
  - [14] N Saad, Hall R L and A B von Keviczky, *J. Math. Phys.* **44** 5021 (2003).
  - [15] Hall R L, N Saad, and A B von Keviczky, *J. Phys. A: Math. Gen.* **34** 1169 (2001).
  - [16] A B von Keviczky, N Saad, and Richard L. Hall, *J. Math. Anal. Appl.* **292** 274 (2004).
  - [17] Hall R L, N Saad, and A B von Keviczky, *J. Math. Phys.* **43** 94 (2002).
  - [18] S Datta, J K Bhattacharjee, *Eur. Phys. J. B* **31** 247 (2003).
  - [19] J K Bhattacharjee, S Bhattacharyya, *J. Phys. A: Math. Gen.* **36** L223 (2003).
  - [20] S K Bandyopadhyay, K Bhattacharyya, *Int. J. Quant. Chem.* **103** 19 (2005).
  - [32] S K Bandyopadhyay, K Bhattacharyya, *Int. J. Quant. Chem.* **106** 390 (2006).
  - [22] Ciftci H, Hall R L and Saad N 2003 *J. Phys. A: Math. Gen.* **36** 11807.
  - [23] Fernández M F 2004 *J. Phys. A: Math. Gen.* **37** 6173.
  - [24] Ciftci H, Hall R L and Saad N 2005 *J. Phys. A: Math. Gen.* **38** 1147.
  - [25] Ciftci H, Hall R L and Saad N 2005 *Phys. Lett. A* **340** 388.
  - [26] Barakat T, Abodayeh K, Mukheimer A 2005 *J. Phys. A: Math. Gen.* **38** 1299.
  - [27] Barakat T 2005 *Phys. Lett. A* **344** 411.
  - [28] Barakat T, Abodayeh K, Abdullah B and Al-Dossary O M 2006 *Canad. J. Phys.* **84** 121-129.
  - [29] Barakat T 2006 *J. Phys. A* **39** 823.
  - [30] Amore P and Fernandez F 2006 *J. Phys. A: Math. Gen.* **39** 10491-10497.
  - [31] Ciftci H, Hall R L, and Saad N 2005 *Phys. Rev. A* **72** 022101.
  - [32] Bayrak O and Boztosun I 2006 *J. Phys. A: Math. Gen.* **39** 6955.
  - [33] Saad N, Hall R L and Ciftci H 2006 *J. Phys. A: Math. Gen.* **39** 8477-8486.
  - [34] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* Dover, New York.



TABLE IV: The energy eigenvalues for Schrödinger's equation (1) with  $\alpha = 4$  and different values of  $\gamma$  for a wide range of values of the coupling  $\lambda$ . For these computations,  $r_0$  varies over the range  $[4, 6.5]$  as  $\lambda$  approaches zero.  $N$  is the number of iterations.

$\alpha = 4$									
$\lambda$	$\gamma$	$E_{AIM}$							N
1000	0	21.369	462	532	163	464	497	98	43
	1	21.522	859	814	112	640	999	87	39
	2	21.827	883	093	646	909	321	84	40
	3	22.281	057	275	014	819	956	75	38
	4	22.877	334	674	778	463	023	90	37
	5	23.610	282	631	878	614	495	76	36
100	0	11.265	080	431	752	838	088	14	68
	1	16.235	741	726	872	875	703	15	69
	2	16.801	763	365	978	787	670	47	67
	3	17.624	891	020	204	476	919	71	65
	4	18.677	251	190	728	170	156	30	63
	5	19.926	538	009	871	777	284	42	58
10	0	6.606	622	512	024	943	661	69	129
	1	7.223	520	393	149	576	761	99	124
	2	8.352	483	528	249	905	501	41	116
	3	9.839	231	320	856	383	508	94	100
	4	11.544	000	451	519	493	138	54	89
	5	13.371	330	123	959	945	355	25	78
1	0	4.494	177	983	369	188			275
	1	5.559	167	225	784	086			246
	2	7.224	287	163	959	573			158
	3	9.108	658	607	516	353			131
	4	11.062	241	719	384	166			107
	5	13.040	015	183	057	043			88
0.1	0	3.575	551	992					260
	1	5.095	284	821					200
	2	7.025	961	149					133
	3	9.011	364	026					90
	4	11.006	336	099					68
	5	13.004	036	433					50
0.01	0	3.205	067	495					450
	1	5.011	917	775					356
	2	7.002	658	316					155
	3	9.001	142	200					85
	4	11.000	634	789					60
	5	13.000	404	001					51
0.001	0	3.068	765						335
	1	5.001	286	52					259
	2	7.000	266	58					88
	3	9.000	114	28					53
	4	11.000	063	49					39
	5	13.000	040	40					33